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NONLINEAR DYNAMICS OF FLEXIBLE STRUCTURES: A FINITE ELEMENT APPROACH

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Abstract—A finite element formulation is presented for analyzing the nonlinear dynamic response of structural systems composed of both rigid bodies and deformable beam elements. The finite strain beam theory proposed by Reissner (1973) is used to model the three-dimensional, fully nonlinear beam element. Finite rotations are described in terms of the modified Rodrigues vector and Euler parameters. Equations of motion governing the system are obtained through the principle of virtual work and the use of relations between angular velocity and rotation parameters. These equations are then written as a system of first-order ordinary differential equations and integrated numerically using a fifth- and sixth-order Runge–Kutta–Verner method. Without the inertia terms, quasi-static motions are analyzed with the aid of a Levenberg–Marquardt algorithm. Numerical examples are presented to illustrate the validity of the proposed method.

1. INTRODUCTION

In order to analyze the nonlinear dynamic response of such modern space structures as satellite antennae, telescopes and orbiting space stations, a method must be used that permits both large rigid body motion and finite deformations of its flexible components. This type of analysis also has applications in the fields of biomechanics and robotics. In this paper a finite element formulation is presented to treat problems which can be modeled as systems composed of both rigid bodies and deformable, finite strain beam elements.

Discussion of past works is limited to three-dimensional dynamic response of flexible multibody systems. Furthermore, only a cursory review of three-dimensional small-strain dynamics of flexible structures is included. Regrettably, much pertinent work, notably that of Simo and Vu-Quoc (1986a–c, 1991), dealing with both two- and three-dimensional static and two-dimensional dynamic response of such systems, is therefore omitted. This is not intended as a comprehensive review of the literature on static and dynamic response of flexible structures, as it is too voluminous to present in the context of this paper.

Numerous authors have considered the dynamic response of flexible beams undergoing large overall motions with the restriction of small elastic deformations or small strains. To incorporate these assumptions a floating frame is introduced with respect to which the gradients of elastic displacements are assumed to be small. The introduction of this floating frame gives rise to Coriolis and centrifugal forces which, even in the case of small elastic deformations, produce nonlinear equations of motion and lead to coupling in the inertia terms. Also, when using methods in which deformations are assumed small *a priori*, it becomes necessary to have a clear understanding of the physical ramifications of those assumptions. For example, Kane *et al.* (1983), when using modal decomposition to analyze a free–free beam undergoing large overall motion, point out that, if one assumes transverse vibrations are small, assuming longitudinal elongation is also small may lead to erroneous results in transverse vibration analysis because of the absence of geometric stiffening effects on the bending behavior of the member. Kane *et al.* (1987) also mention various important beam effects which are frequently neglected in order to enhance computational efficiency or to model specific types of problems. A finite element approach to deal with systems composed of both rigid bodies and flexible beam members was presented by Belytschko *et al.* (1977) for the case of small deformations in flexible elements.

Dynamic response of systems of flexible beams undergoing finite strains has been considered by Simo (1985), Simo and Vu-Quoc (1988), Iura and Atluri (1988), and Cardona and Geradin (1988). Vu-Quoc (1986) has also considered these systems and has alluded to their solution with the inclusion of rigid bodies. In all of these formulations the use of a floating frame was avoided by using a fully nonlinear beam theory and referring all variables to the inertial frame. The approach used entailed linearizing the equations of motion, solving for the incremental variables using an implicit integration technique, and then employing a particular updating algorithm to find the current system configuration. It is important to note that in each of these formulations the components of the rotation vector itself were used to describe the finite rotations. More recent works by Park *et al.* (1991) and Downer *et al.* (1992) deal with the dynamics of rigid–flexible multibody systems in detail. While the rigid body kinematics in their work is the same as in this paper, the present treatment of flexible beam elements employs total strain and total stress formulation rather than the incremental strain and incremental stress formulation of these two references.

In the formulation used in this paper, the translational motion of the system is described by the displacement and velocity of the nodes and the rotational motion by the nodal angular velocity and rotation parameters. Relations between angular velocity and rotation parameters allow for computation of the rotation matrix at each time step without any matrix multiplication. As pointed out in Simo and Vu-Quoc (1986c, 1988), the equations of motion can be equivalently referred to either the deformed (spatial) or undeformed (material, initial) configuration. In what follows the rotational equations of motion are written with respect to the deformed configuration, while translational equations are referred to the original configuration. By using approximate diagonal forms of the mass and inertia matrices, this formulation produces an inertia matrix which is uncoupled and has a very simple form; one in which rigid body contributions to the inertia terms are constants.

The Reissner beam theory, with the inclusion of shear strains, is much more general than the Euler theory and can be used to efficiently model systems where shear behavior is important. It is also useful because the orientation of the beam cross-section is explicitly defined at all times, making it easy to incorporate follower forces into the loadings.

The equations of motion in finite element form are integrated explicitly to yield nodal values of displacement, velocity, rotation parameters and angular velocity at each time step. Once these are obtained, simple relations yield values for strains, curvatures, internal stress resultants, and the rotation matrix. This procedure, with a finite strain beam model, imposes no limitations on the magnitudes or time histories of loadings and deformations.

2. KINEMATICS

The structural system considered here is made up of rigid bodies and flexible beam members. Consider a single structural component made up of two rigid bodies, serving as nodes connected by a flexible beam element, as shown in Fig. 1.

Simple rotation

Consider first the description of the rotation of a rigid body by Kane *et al.* (1983). A motion of a rigid body or reference frame B relative to a rigid body or reference frame A is called a simple rotation of B in A, if there exists a line L, called an axis of rotation, whose orientation relative to both A and B remains unchanged throughout the motion. Any change in the relative orientation of A and B can be produced by means of a simple rotation of B in A. This simple rotation can be expressed by specifying a unit vector $\boldsymbol{\lambda}$ parallel to L, and the radian measure of an angle θ describing the magnitude of the rotation about $\boldsymbol{\lambda}$.

System description

Here and in what follows, the subscript n denotes the nth node, subscripts i and j denote the first and second nodes of a single structural component, respectively, and $\alpha = 1,2,3$.

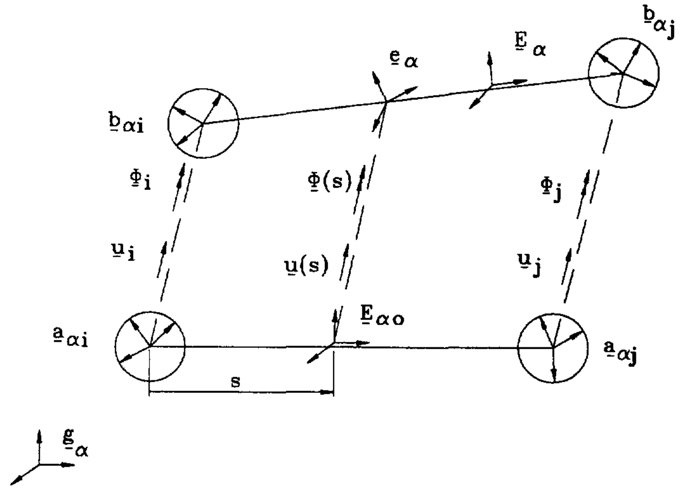

Fig. 1. Single structural component.

Consider again Fig. 1, and let a structural component be in its reference configuration at time $t = t_0$, relative to a fixed unit triad $\mathbf{g}_\alpha$ describing an inertial frame G. Let $\mathbf{a}_{\alpha n}$ be a unit triad fixed in a frame A describing the initial orientation of a node, and $\mathbf{E}_{\alpha 0}$ describe the initial orientation of both the beam cross-section and the beam axis. At a later time t, the component moves to a new configuration in which the orientation of the node is described by a unit triad $\mathbf{b}_{\alpha n}$, fixed in a frame B and initially equal to the triad $\mathbf{a}_{\alpha n}$. The new orientation of the beam cross-section is described by the unit triad $\mathbf{e}_\alpha$ and the beam axis by $\mathbf{E}_\alpha$, both initially equal to $\mathbf{E}_{\alpha 0}$. The relative orientations of $\mathbf{a}_{\alpha n}$ and $\mathbf{b}_{\alpha n}$, as well as $\mathbf{E}_{\alpha 0}$ and $\mathbf{e}_\alpha$, can be described by simple rotations, while $\mathbf{E}_\alpha$ and $\mathbf{E}_{\alpha 0}$ are related through a simple coordinate transformation.

In the finite element model, the component configuration is completely described by the position and orientation of the nodes. The position vector of node n at any time t will be described by its original position vector

$$\mathbf{q}_n = q_{1n}\mathbf{g}_1 + q_{2n}\mathbf{g}_2 + q_{3n}\mathbf{g}_3 \tag{1}$$

and its displacement vector

$$\mathbf{u}_n = u_{1n}\mathbf{a}_{1n} + u_{2n}\mathbf{a}_{2n} + u_{3n}\mathbf{a}_{3n}. \tag{2}$$

The orientation of the node will be described by its original orientation matrix $\mathbf{C}_n^0$ and its rotation matrix $\mathbf{C}_n$, the components of which are given in Appendix A. The elements of the rotation matrix are time dependent functions of the rotation parameters which, in turn, are functions of the nodal angular velocity

$$\omega_n = \omega_{1n}\mathbf{b}_{1n} + \omega_{2n}\mathbf{b}_{2n} + \omega_{3n}\mathbf{b}_{3n}. \tag{3}$$

Rotation parameters and angular velocity

In the following it is understood that we are discussing the behavior of a single node. Therefore, the subscript n will be suppressed.

The manner in which the simple rotation of the nodes is treated is of considerable interest because of its implications on the nature of the governing equations as well as the form of the inertia matrix. Three-dimensional finite rotations are not vector functions and therefore do not take on desirable qualities of vector algebra, such as commutativity in addition, or vector calculus, such as differentiation in time.

There are however, several vectors associated with the unit vector $\boldsymbol{\lambda}$ and the angle θ that not only have these qualities, but are also directly related to angular velocity (Kane *et* *al.*, 1983). One of these is the Euler vector $\boldsymbol{\varepsilon}$, and four scalar quantities, $\varepsilon_1, \ldots, \varepsilon_4$, that can describe a finite rotation of a node by letting

$$\boldsymbol{\varepsilon} \equiv \sin\frac{\theta}{2}\boldsymbol{\lambda} \tag{4}$$

$$\varepsilon_\alpha \equiv \boldsymbol{\varepsilon}\cdot\mathbf{a}_\alpha = \boldsymbol{\varepsilon}\cdot\mathbf{b}_\alpha \quad (\alpha = 1, 2, 3) \tag{5}$$

and

$$\varepsilon_4 \equiv \cos\frac{\theta}{2}. \tag{6}$$

The second of these is the modified Rodrigues vector $\boldsymbol{\phi}$, and its components ϕ_1, ϕ_2 and ϕ_3, that are defined by

$$\boldsymbol{\phi} \equiv 2\tan\frac{\theta}{2}\boldsymbol{\lambda} \tag{7}$$

and

$$\phi_\alpha \equiv \boldsymbol{\phi}\cdot\mathbf{a}_\alpha = \boldsymbol{\phi}\cdot\mathbf{b}_\alpha \quad (\alpha = 1, 2, 3). \tag{8}$$

It should be noted that the rotation parameters introduced here have the same components with respect to both the initial and current configurations.

In order to compute the rotation parameters and through them the rotation matrix, a relation between these parameters and the angular velocity of the nodes must be introduced. Using the Euler parameters these equations are

$$\boldsymbol{\omega} = 2\left(\varepsilon_4\frac{{}^B\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} - \dot{\varepsilon}_4\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\frac{{}^B\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t}\right) \tag{9}$$

$$\frac{{}^B\mathrm{d}\boldsymbol{\varepsilon}}{\mathrm{d}t} = \tfrac{1}{2}(\varepsilon_4\boldsymbol{\omega} + \boldsymbol{\varepsilon}\times\boldsymbol{\omega}) \tag{10}$$

and

$$\dot{\varepsilon}_4 = -\tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\varepsilon}. \tag{11}$$

In terms of the modified Rodrigues vector we have

$$\boldsymbol{\omega} = \frac{1}{1+\frac{1}{4}\phi^2}\left(\frac{{}^B\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} - \tfrac{1}{2}\boldsymbol{\phi}\times\frac{{}^B\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t}\right) \tag{12}$$

and

$$\frac{{}^B\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}t} = \boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\omega} + \tfrac{1}{4}\boldsymbol{\phi}\boldsymbol{\phi}\cdot\boldsymbol{\omega}, \tag{13}$$

where $(\dot{\ })$ denotes time differentiation and ${}^B\mathrm{d}/\mathrm{d}t$ denotes time differentiation in frame B.

At each time step, the numerical integration of eqns (10) and (11), if Euler parameters are used, or eqn (13) if modified Rodrigues parameters are used, will yield values of these parameters, which can be used to compute the nodal rotation matrix.

3. BEAM EQUATIONS

The finite strain beam element is due to Reissner (1973). The parameter s is used to denote the distance along the beam axis and the length of the beam at time $t = t_0$ is l_0. Both curvatures and force strain components are given with respect to the deformed configuration. That is,

$$\boldsymbol{\chi} = \chi_1 \mathbf{e}_1 + \chi_2 \mathbf{e}_2 + \chi_3 \mathbf{e}_3 \tag{14}$$

and

$$\boldsymbol{\gamma} = \gamma_1 \mathbf{e}_1 + \gamma_2 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3. \tag{15}$$

Curvatures

The curvatures are functions only of the rotation parameters, and are shown here in vector form:

$$\boldsymbol{\chi} = \frac{1}{1+\frac{1}{4}\phi^2}\left(\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}s} - \tfrac{1}{2}\boldsymbol{\phi} \times \frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}s}\right). \tag{16}$$

Strains

The strains depend not only on the displacements but also on the rotation parameters which, as detailed in Appendix A, define the elements of the cross-section rotation matrix $\mathbf{C}_e$. These strains are shown here as a 3×1 column vector:

$$\boldsymbol{\gamma} = \mathbf{C}_e \begin{Bmatrix} u_1' \\ u_2' \\ u_3' \end{Bmatrix} + \begin{Bmatrix} C_e^{11} - 1 \\ C_e^{21} \\ C_e^{31} \end{Bmatrix}, \tag{17}$$

where ( )′ denotes differentiation with respect to the spatial parameter s.

Constitutive equations

The constitutive model used is taken from Euler beam theory. Letting $\mathbf{m}$ denote the components of the internal moment vector and $\mathbf{p}$ denote the components of the force vector, both with respect to $\mathbf{e}$, we have

$$\{\mathbf{p}\} = \lceil EA \quad GA \quad GA \rfloor \{\boldsymbol{\gamma}\} \tag{18}$$

and

$$\{\mathbf{m}\} = \lceil GJ \quad EI_2 \quad EI_3 \rfloor \{\boldsymbol{\chi}\}, \tag{19}$$

where $\lceil \ldots \rfloor$ denotes a diagonal matrix with the elements shown. To obtain more accurate constitutive models one would have to perform some large strain experiments on the desired material.

It is important to note here that, by modifying the quantities in eqn (18), one can effectively model different types of beam cross-sections. For example, if an I-beam were being modeled, the area of the cross-section would be used in the first term, while the area of the web would be used in the second two terms. This ability to adjust the parameters describing the shear behavior becomes particularly important in cases such as the modeling of deep beams, which cannot be accomplished using the Euler theory.

4. VIRTUAL WORK

Already having eqns (10) and (11), or (13), the remaining equations of motion are generated using the principle of virtual work. Letting the virtual work of the external forces minus the virtual work of inertia forces equal the internal virtual work of the beam elements, we have

$$\sum (W_{\text{ext}} - W^{\text{n}}_{\text{inertia}} - W^{\text{b}}_{\text{inertia}}) = \sum W_{\text{int}} \tag{20}$$

in which superscripts n and b denote nodes and beams, respectively and Σ denotes the summation over both.

Virtual displacements and rotations

The virtual displacements of the nodes will be $\delta\mathbf{u}_n$ and the virtual rotations will be $\delta\boldsymbol{\theta}_n$. When discussing a single structural component the virtual displacements of nodes i and j will be $\delta\mathbf{u}_i$ and $\delta\mathbf{u}_j$ and the virtual rotations will be $\delta\boldsymbol{\theta}_i$ and $\delta\boldsymbol{\theta}_j$. The virtual displacements and rotations at a beam cross-section described by the parameter s will be interpolated from the nodal values and will be $\delta\mathbf{u}$ and $\delta\boldsymbol{\theta}$. In order to present the equations in matrix form, we will introduce four column vectors to describe the variations. For a single node, we will use the 3×1 vectors

$$\{\delta\mathbf{u}_n\} \tag{21}$$

and

$$\{\delta\boldsymbol{\theta}_n\}, \tag{22}$$

while for the two nodes of a structural component, we choose instead the 6×1 vectors

$$\{\delta\mathbf{u}_{\text{e}}\} = \begin{Bmatrix} \delta\mathbf{u}_i \\ \delta\mathbf{u}_j \end{Bmatrix} \tag{23}$$

and

$$\{\delta\boldsymbol{\theta}_{\text{e}}\} = \begin{Bmatrix} \delta\boldsymbol{\theta}_i \\ \delta\boldsymbol{\theta}_j \end{Bmatrix}. \tag{24}$$

Using node n as an example, we note that the virtual displacements are written with respect to the original configuration, and the virtual rotations are written with respect to the deformed configuration:

$$\delta\mathbf{u}_n = \delta u_{1n}\mathbf{a}_{1n} + \delta u_{2n}\mathbf{a}_{2n} + \delta u_{3n}\mathbf{a}_{3n} \tag{25}$$

$$\delta\boldsymbol{\theta}_n = \delta\theta_{1n}\mathbf{b}_{1n} + \delta\theta_{2n}\mathbf{b}_{2n} + \delta\theta_{3n}\mathbf{b}_{3n}. \tag{26}$$

External forces and moments

The virtual work done by the external forces and moments of a node will be

$$W_{\text{ext}} = \mathbf{P}_n \cdot \delta\mathbf{u}_n + \mathbf{M}_n \cdot \delta\boldsymbol{\theta}_n. \tag{27}$$

The external forces and moments must correspond to the virtual displacements and rotations. Thus, we must have

$$\mathbf{P}_n = P_{1n}\mathbf{a}_{1n} + P_{2n}\mathbf{a}_{2n} + P_{3n}\mathbf{a}_{3n} \tag{28}$$

and

$$\mathbf{M}_n = M_{1n}\mathbf{b}_{1n} + M_{2n}\mathbf{b}_{2n} + M_{3n}\mathbf{b}_{3n}. \tag{29}$$

Again we introduce two 3×1 column vectors:

$$\{\mathbf{P}_n\} \tag{30}$$

and

$$\{\mathbf{M}_n\} \tag{31}$$

Using eqns (21), (22), (30) and (31) we can rewrite (27) in matrix form:

$$W_{\text{ext}} = \{\delta \mathbf{u}_n\}^{\mathrm{T}} \{\mathbf{P}_n\} + \{\delta \boldsymbol{\theta}_n\}^{\mathrm{T}} \{\mathbf{M}_n\}. \tag{32}$$

Inertia of the nodes

To express the virtual work done by the inertia forces of the nodes, we will first introduce the following matrices. Letting m_n be the mass of node n we have the nodal 3×3 mass matrix

$$\mathbf{m}_n = \lceil m_n \quad m_n \quad m_n \rfloor. \tag{33}$$

By making the principal axes of the rigid body correspond to the triads $\mathbf{a}_n$ and $\mathbf{b}_n$ at time $t = t_0$ the rotatory inertia matrix will also be a 3×3 diagonal matrix:

$$\mathbf{I}_n = \lceil I_{11} \quad I_{22} \quad I_{33} \rfloor. \tag{34}$$

We also introduce the 3×1 column vectors of angular acceleration

$$\{\dot{\boldsymbol{\omega}}_n\} \tag{35}$$

and translational acceleration

$$\{\ddot{\mathbf{u}}_n\}, \tag{36}$$

where $(\dot{\ })$ in eqn (35) denotes time differentiation in frame B or A, and $(\ddot{\ })$ in eqn (36) denotes time differentiation twice in frame A. Using these definitions, the virtual work of the inertial forces of the nodes is

$$W^{\mathrm{n}}_{\text{inertia}} = {}^{\mathrm{t}}W^{\mathrm{n}}_{\text{inertia}} + {}^{\mathrm{r}}W^{\mathrm{n}}_{\text{inertia}}, \tag{37}$$

where

$${}^{\mathrm{t}}W^{\mathrm{n}}_{\text{inertia}} = \{\delta \mathbf{u}_n\}^{\mathrm{T}} \lceil \mathbf{m}_n \rfloor \{\ddot{\mathbf{u}}_n\} \tag{38}$$

and

$${}^{\mathrm{r}}W^{\mathrm{n}}_{\text{inertia}} = \{\delta \boldsymbol{\theta}_n\}^{\mathrm{T}} (\lceil \mathbf{I}_n \rfloor \{\dot{\boldsymbol{\omega}}_n\} + \{\boldsymbol{\Xi}^{\mathrm{n}}\}), \tag{39}$$

where

$$\{\boldsymbol{\Xi}^{\mathrm{n}}\} = \begin{Bmatrix} \omega_{2n}\omega_{3n}(I_{33} - I_{22}) \\ \omega_{1n}\omega_{3n}(I_{11} - I_{33}) \\ \omega_{1n}\omega_{2n}(I_{22} - I_{11}) \end{Bmatrix}. \tag{40}$$

Note that the translational inertia is expressed in terms of vectors fixed in the original configuration, while the rotatory inertia is in terms of vectors fixed in the deformed configuration. It is also important to note that, because of the formulation used, the nodal mass and inertia matrices are constant and diagonal in time.

Inertia of the beam

The virtual work done by the inertia forces of the beam is

$$W^{b}_{inertia} = {}^{t}W^{b}_{inertia} + {}^{r}W^{b}_{inertia} \tag{41}$$

where

$${}^{t}W^{b}_{inertia} = \int_0^{l_0} \rho_0 A_0 \{\ddot{\mathbf{u}}\}\{\delta\mathbf{u}\}\, \mathrm{d}s \tag{42}$$

and

$${}^{r}W^{b}_{inertia} = \int_0^{l_0} \rho_0 \{\delta\boldsymbol{\theta}\}^{T} [\tilde{I}] \{\dot{\omega}\}\, \mathrm{d}s. \tag{43}$$

The acceleration and virtual displacement of the beam centerline are $\{\ddot{\mathbf{u}}\}$ and $\{\delta\mathbf{u}\}$, respectively. The column vector $\{\delta\boldsymbol{\theta}\}^{T}$ contains the three components of the virtual rotation of the beam cross-section and $[\tilde{I}]$ is the matrix of section inertia properties.

Internal forces and moments

The virtual work done by the internal forces and moments of one component will be

$$W_{int} = {}^{t}W_{int} + {}^{r}W_{int}, \tag{44}$$

where

$${}^{t}W_{int} = \int_0^{l_0} \mathbf{p} \cdot \delta\gamma\, \mathrm{d}s, \tag{45}$$

and

$${}^{r}W_{int} = \int_0^{l_0} \mathbf{m} \cdot \delta\chi\, \mathrm{d}s. \tag{46}$$

The relations between the virtual strains and the virtual displacements are given by Reissner (1973) and are as follows:

$$\delta\chi = \delta\theta' \tag{47}$$

and

$$\delta\gamma = \delta\mathbf{u}' + \mathbf{R}' \times \delta\boldsymbol{\theta}, \tag{48}$$

where

$$\mathbf{R}' = \mathbf{E}_{10} + \mathbf{u}' = (1+\gamma_1)\mathbf{e}_1 + \gamma_2\mathbf{e}_2 + \gamma_0\mathbf{e}_3 \tag{49}$$

and $(\ )'$ denotes differentiation with respect to the spatial parameter s.

5. FINITE ELEMENT FORMULATION

In this formulation the system's configuration will be described by the kinematics of the nodes. In addition to the definitions of the nodal displacements and the nodal virtual displacements and rotations, we now introduce the nodal rotation parameter vector,

$$\boldsymbol{\phi}_n = \phi_{1n}\mathbf{b}_{1n} + \phi_{2n}\mathbf{b}_{2n} + \phi_{3n}\mathbf{b}_{3n}, \tag{50}$$

or

$$\boldsymbol{\phi}_n = \phi_{1n}\mathbf{a}_{1n} + \phi_{2n}\mathbf{a}_{2n} + \phi_{3n}\mathbf{a}_{3n} \tag{51}$$

and in matrix form the 3×1 column vector

$$\{\boldsymbol{\phi}_n\}. \tag{52}$$

Note that the components of the modified Rodrigues vector are the same when written in frames A or B.

Discretization in space

The values describing the position and orientation of the beam element are interpolated from the values at the nodes. In the following, superscripts e and E_0 will denote frames in which $\mathbf{e}_\alpha$ and $\mathbf{E}_{\alpha 0}$ are fixed. $\mathbf{N}$ will represent the interpolation matrix and the following functions will be employed:

$$^{E_0}\{\mathbf{u}\} = \mathbf{NT}\{\mathbf{u}_e\} \tag{53}$$

$$^{E_0}\{\delta\mathbf{u}\} = \mathbf{NT}\{\delta\mathbf{u}_e\} \tag{54}$$

$$^{e,E_0}\{\boldsymbol{\phi}\} = \mathbf{NT}\{\boldsymbol{\phi}_e\} \tag{55}$$

$$^{e}\{\delta\boldsymbol{\theta}\} = \mathbf{C}_e\mathbf{NTC}^{\mathrm{T}}\{\delta\boldsymbol{\theta}_e\}, \tag{56}$$

where

$$\{\mathbf{u}_e\} = \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{Bmatrix} \tag{57}$$

$$\{\boldsymbol{\phi}_e\} = \begin{Bmatrix} \boldsymbol{\phi}_i \\ \boldsymbol{\phi}_j \end{Bmatrix} \tag{58}$$

$$\mathbf{C} = \lceil \mathbf{C}_i \quad \mathbf{C}_j \rfloor \tag{59}$$

$$\mathbf{T} = \lceil \mathbf{T}_i \quad \mathbf{T}_j \rfloor \tag{60}$$

and $\mathbf{C}_i$, $\mathbf{C}_j$, $\mathbf{T}_i$ and $\mathbf{T}_j$ are defined in Appendix A.

We will use a 3×6 linear interpolation matrix as follows:

$$\mathbf{N} = \begin{bmatrix} 1-s/l_0 & 0 & 0 & s/l_0 & 0 & 0 \\ 0 & 1-s/l_0 & 0 & 0 & s/l_0 & 0 \\ 0 & 0 & 1-s/l_0 & 0 & 0 & s/l_0 \end{bmatrix}. \tag{61}$$

Through the use of these functions, the entire system configuration can be determined once the positions and orientations of the nodes are known.

Inertia properties

In this formulation, diagonal forms of both the mass and inertia matrices are used. The approximation of each element's contribution to the inertia matrix is taken directly from Belytchko *et al.* (1977). To each node an element contributes:

(1) half of its translational mass;
(2) half of its mass moment of inertia about the beam's axis E_1;
(3) half of the mass moments about E_2 and E_3, which are $\frac{1}{12}ml_0^2$, where l_0 is the original length of the element.

Inertia of the beam

Using these approximations and letting m_b be the mass of the beam element, the contribution to each of the nodal mass matrices $\mathbf{m}_i$ and $\mathbf{m}_j$ will be

$$\mathbf{m}_i = \mathbf{m}_j = \lceil \tfrac{1}{2}m_b \quad \tfrac{1}{2}m_b \quad \tfrac{1}{2}m_b \rfloor. \tag{62}$$

The contribution of the beam inertia matrix to each of the nodes is constant when written in terms of the triad $\mathbf{E}_\alpha$, which describes the current orientation of the beam element's centerline. The contribution to the inertia matrix of nodes i and j in terms of these unit vectors is

$$^{E}\mathbf{I}_i = {}^{E}\mathbf{I}_j = \lceil \tfrac{1}{2}I_{11} \quad \tfrac{1}{24}ml_0^2 \quad \tfrac{1}{24}ml_0^2 \rfloor. \tag{63}$$

The equations of motion, however, are written in terms of vectors fixed in the nodes. Therefore, this inertia matrix must be transformed to correspond to the triads $\mathbf{b}_{\alpha i}$ and $\mathbf{b}_{\alpha j}$, so that the contributions can be added to $\lceil \mathbf{I}_i \rfloor$.

Again referring to Appendix A for the definitions of the orthogonal transformation matrices, the contribution of the beam inertia to node i in terms of $\mathbf{b}_i$ is

$$^{B_i}\mathbf{I}_i = \mathbf{D}_i\, {}^{E}\mathbf{I}_i \mathbf{D}_i^{\mathrm{T}} \tag{64}$$

and the contribution to node j in terms of $\mathbf{b}_j$ is

$$^{B_j}\mathbf{I}_j = \mathbf{D}_j\, {}^{E}\mathbf{I}_j \mathbf{D}_j^{\mathrm{T}}, \tag{65}$$

where B_i and B_j refer to frames fixed in each of the respective nodes.

Using these definitions, the virtual work of the inertia forces of the beam can be written

$$W^{\mathrm{b}}_{\text{inertia}} = {}^{\mathrm{t}}W^{\mathrm{b}}_{\text{inertia}} + {}^{\mathrm{r}}W^{\mathrm{b}}_{\text{inertia}}, \tag{66}$$

where

$$^{t}W^{\mathrm{b}}_{\text{inertia}} = \{\delta\mathbf{u}_e\}^{\mathrm{T}} \lceil \mathbf{m}_i, \quad \mathbf{m}_j \rfloor \begin{Bmatrix} \ddot{\mathbf{u}}_i \\ \ddot{\mathbf{u}}_j \end{Bmatrix}, \tag{67}$$

$$^{r}W^{\mathrm{b}}_{\text{inertia}} = \{\delta\boldsymbol{\theta}_e\}^{\mathrm{T}} \left(\lceil {}^{B_i}\mathbf{I}_i, \quad {}^{B_j}\mathbf{I}_j \rfloor \begin{Bmatrix} \dot{\boldsymbol{\omega}}_i \\ \dot{\boldsymbol{\omega}}_j \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{\Xi}_i^{\mathrm{b}} \\ \boldsymbol{\Xi}_j^{\mathrm{b}} \end{Bmatrix} \right) \tag{68}$$

and $\{\Xi^{\mathrm{b}}\}$ for a single node is of the form:

$$\begin{Bmatrix} \omega_2\omega_3(\tilde{I}_{33} - \tilde{I}_{22}) + \omega_1\omega_2\tilde{I}_{13} - \omega_3^2\tilde{I}_{23} - \omega_1\omega_3\tilde{I}_{12} + \omega_2^2\tilde{I}_{32} \\ \omega_1\omega_3(\tilde{I}_{11} - \tilde{I}_{33}) - \omega_1\omega_2\tilde{I}_{32} + \omega_3^2\tilde{I}_{13} + \omega_2\omega_3\tilde{I}_{21} - \omega_1^2\tilde{I}_{31} \\ \omega_1\omega_2(\tilde{I}_{22} - \tilde{I}_{11}) + \omega_1\omega_3\tilde{I}_{32} - \omega_2^2\tilde{I}_{12} - \omega_2\omega_3\tilde{I}_{31} + \omega_1^2\tilde{I}_{21} \end{Bmatrix}. \tag{69}$$

In the above expression the subscripts describing the particular node and superscripts describing the frame have been supressed. The quantities $\tilde{I}_{\alpha\beta}$ describe elements of the inertia matrices $^{B_i}\mathbf{I}_i$ or $^{B_j}\mathbf{I}_j$. Note that the contribution of the beam inertia matrix to the assembled nodal inertia matrix will be full and configuration dependent. Also, due to time dependence of the inertia matrix we see additional terms appear in the $\{\Xi^b\}$ vector which were not present in the $\{\Xi^n\}$ vector.

Internal forces and moments

From eqns (14), (15), (18) and (19), it is shown that the components of **p** and **m** may be written in terms of **e**. Because the virtual strains and curvatures are functions of the virtual displacements and rotations, which are interpolated from the nodal values of these quantities, the strains are first written in terms of the triads $\mathbf{a}_{\alpha i}$, $\mathbf{a}_{\alpha j}$, $\mathbf{b}_{\alpha i}$ and $\mathbf{b}_{\alpha j}$.

Let us now introduce two 6×1 column vectors $\{{}^t\mathbf{Q}_b\}$ and $\{{}^r\mathbf{Q}_b\}$, which we will call the generalized internal force and moment vectors such that

$$W_{int} = \{\delta\mathbf{u}_e\}^T\{{}^t\mathbf{Q}_b\} + \{\delta\boldsymbol{\theta}_e\}^T\{{}^r\mathbf{Q}_b\}. \tag{70}$$

In order to evaluate the dot products in eqns (45) and (46), the components of the internal forces and moments and the virtual strains must be written with respect to the same base vectors. The transformations necessary to do this are such that the generalized internal force and moment vectors needed for the equations of motion are highly nonlinear functions of the displacements, the rotation parameters, and their derivatives with respect to s.

The values of $\{{}^t\mathbf{Q}_b\}$ and $\{{}^r\mathbf{Q}_b\}$ are determined by evaluating the integrands in eqns (45) and (46) with the symbolic computer package *Mathematica* (1988–1991) and then integrating them numerically with a one-point Gauss method to avoid shear locking.

Equations of motion

Substituting eqns (32), (37)–(39), (66)–(68), and (70) into eqn (20), we get the equations of motion for a single node:

$$[\mathbf{M}]\{\ddot{\mathbf{u}}_n\} + \{{}^t\mathbf{Q}_n\} = \{\mathbf{F}_n\} \tag{71}$$

and

$$[\mathbf{I}]\{\dot{\boldsymbol{\omega}}_n\} + \{\Xi^n\} + \{\Xi^b\} + \{{}^r\mathbf{Q}_n\} = \{\mathbf{M}_n\} \tag{72}$$

where

$$[\mathbf{M}] = \sum([\mathbf{m}_n] + [\mathbf{m}_b]) \tag{73}$$

$$[\mathbf{I}] = \sum([\mathbf{I}_n] + [\mathbf{I}_b]) \tag{74}$$

and $[\mathbf{m}_n]$ and $[\mathbf{I}_n]$ denote contributions of the nodes to the nodal mass and inertia matrices while $[\mathbf{m}_b]$ and $[\mathbf{I}_b]$ denote contributions of the beam elements attached to a particular node. Also, $\{{}^t\mathbf{Q}_n\}$ and $\{{}^r\mathbf{Q}_n\}$ are 3×1 column vectors describing the contributions of $\{{}^t\mathbf{Q}_b\}$ and $\{{}^r\mathbf{Q}_b\}$ to a particular node. Here Σ denotes summation over all beam elements attached to the node.

6. INTEGRATION IN TIME

By writing eqns (13), (71) and (72) for each node, we produce equations governing the motion of the entire structural system. These equations are then written as a system of first-order ordinary differential equations of the following form:

$$\{\dot{\mathbf{u}}\} = \{\mathbf{v}\} \tag{75}$$

$$\{\dot{\mathbf{v}}\} \equiv f(\boldsymbol{\phi}, \mathbf{u}, t) \tag{76}$$

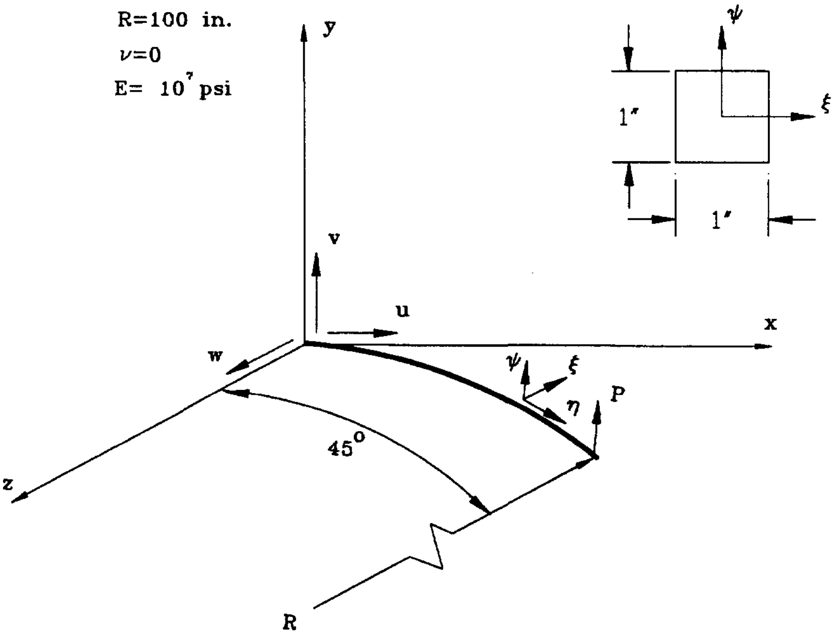

Fig. 2. Cantilever 45° bend.

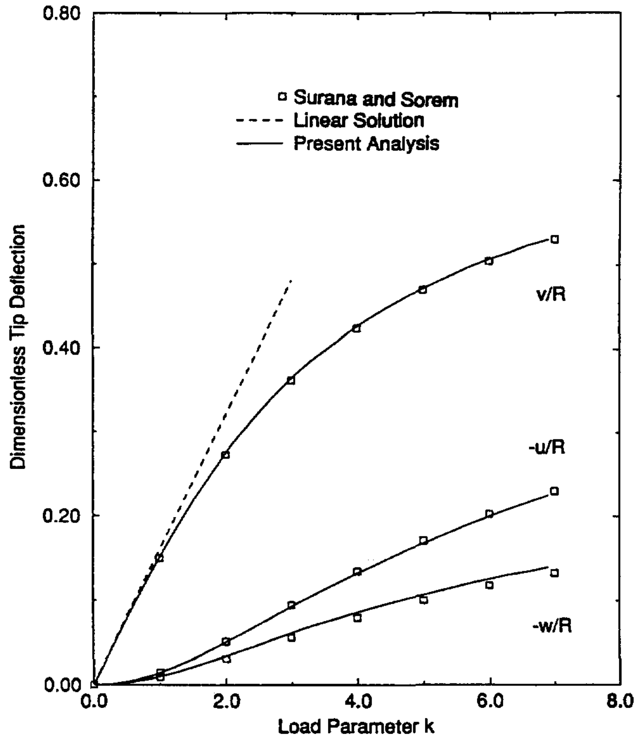

Fig. 3. Tip deflections of 45° bend.

$$\{\dot{\omega}\} \equiv f(\phi, \omega, \mathbf{u}, t) \tag{77}$$

$$\{\dot{\phi}\} \equiv f(\phi, \omega), \tag{78}$$

where the functions in eqns (76)–(78) are defined in eqns (71), (72) and (13). These equations are integrated numerically by means of subroutine DIVPRK of IMSL (1991a), which

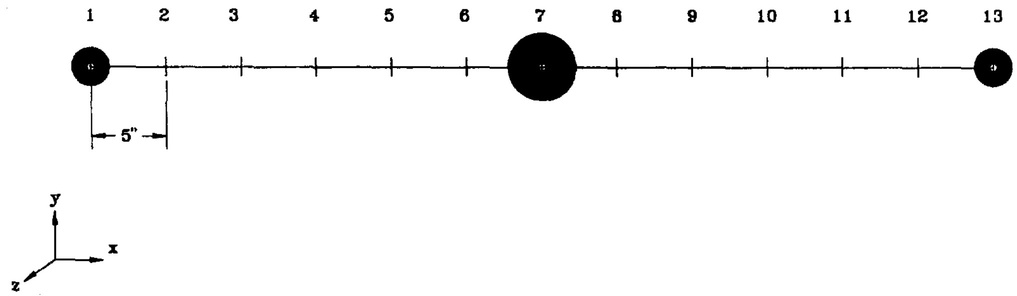

Fig. 4. Rigid hub with flexible appendages.

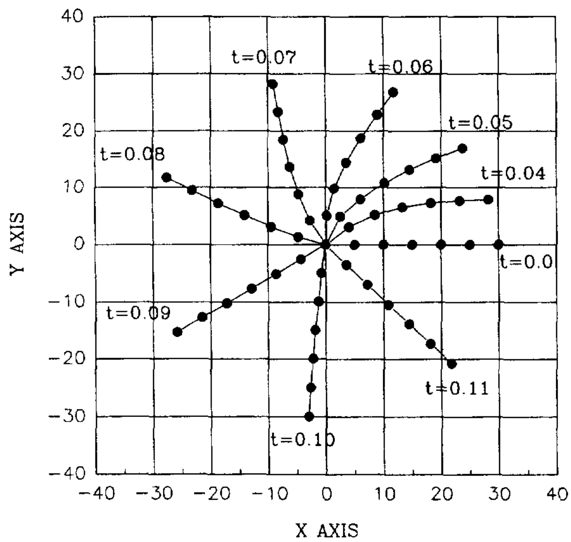

Fig. 5. Projections of displacements on x–y plane.

uses the fifth- and sixth-order Runge–Kutta–Verner method with double precision. If the dynamic terms are eliminated, the static equations are solved using subroutine DNEQNF of IMSL (1991), which uses a Levenberg–Marquardt algorithm with double precision and a finite difference approximation to the Jacobian.

Singularity in the rotation parameters

Note that there is a singularity present at $\theta = \pi$ when using the modified Rodrigues parameters. In order to avoid this singularity, one can replace eqns (76)–(78) with the corresponding equations in terms of Euler parameters. The changes include using eqns (10) and (11) instead of eqn (13), and writing the curvatures and transformation matrices in terms of Euler parameters.

The angle at which the modified Rodrigues parameters become singular corresponds to $\varepsilon_4 = 0$ when using the Euler parameters. If one wishes, the singularity can be dealt with by using modified Rodrigues parameters away from this point, keeping track of ε_4, and switching to Euler parameters within a certain vicinity of the singularity.

7. NUMERICAL RESULTS

Example 1. *Cantilever* 45° *bend with tip load*

The 45° bend shown in Fig. 2 is curved in the x–z plane and subjected to a concentrated tip load in the y-direction. The bend is fixed at $(0, 0, 0)$ and has an average radius of 100 in and a cross-sectional area of 1 in^2. Nine equal beam elements were used to model the problem and a static solution was obtained.

This problem has been examined by Surana and Sorem (1989), Bathe and Bolourchi (1979), Hsiao *et al.* (1987), and several others. The results of the present analysis are shown in Fig. 3 and are compared with the results given in Surana and Sorem (1989). Note that the load parameter $k = PR^2/EI$.

Example 2. *Rigid hub with flexible appendages*

A "spin-up" problem similar to that in Kane *et al.* (1983) is examined here with the inclusion of flexible appendages. The structure shown in Fig. 4 has a rigid body at node 7 with principal moments of inertia $I_1 = I_2 = 2$ in^4 and $I_3 = 1$ in^4 and concentrated masses of 0.05 pounds at nodes 1 and 13. The cantilevers are six equal steel members with circular cross-sections and 0.5 in radii. Node 7 is subjected to constant moments of 1000 in kips about the axis defined by its $\mathbf{b}_3$ unit vector and 5 in kips about its $\mathbf{b}_1$ axis. The projections of the displacements on the x–y plane are shown at several times in Fig. 5.

REFERENCES

Bathe, K.-J. and Bolourchi, S. (1979). Large displacement analysis of three-dimensional beam structures. *Int. J. Numer. Meth. Engng* **14,** 961–986.

Belytschko, T., Schwer, L. and Klein, M. J. (1977). Large displacement, transient analysis of space frames. *Int. J. Numer. Meth. Engng* **11,** 65–84.

Cardona, A. and Geradin, M. (1988). A beam finite element non-linear theory with finite rotations. *Int. J. Numer. Meth. Engng* **26,** 2403–2438.

Downer, J. D., Park, K. C. and Chiou, J. C. (1992). Dynamics of flexible beams for multibody systems: a computational procedure. *Comput. Meth. Appl. Mech. Engng* **96,** 373–408.

Hsiao, K.-M., Horng, H.-J. and Chen, Y.-R. (1987). A corotational procedure that handles large rotations of spatial structures. *Comput. Struct.* **27**(6), 769–781.

IMSL Math/Library DIVPRK Subroutine (1991a). 2.0, pp. 747–754.

IMSL Math/Library DNEQNF Subroutine (1991b). 2.0, pp. 946–948.

Iura, M. and Atluri, S. N. (1988). Dynamic analysis of finitely stretched and rotated three-dimensional space-curved beams. *Comput. Struct.* **29**(5), 875–889.

Kane, T. R., Likins, P. W. and Levinson, D. A. (1983). *Spacecraft Dynamics*. McGraw-Hill, New York.

Kane, T. R., Ryan, R. R. and Banerjee, A. K. (1987). Dynamics of a cantilever beam attached to a moving base. *J. Guidance* **10**(2), 139–151.

Mathematica 2.0 for IBM RISC System 6000 (1988–1991). Wolfram Research Inc.

Park, K. C., Downer, J. D., Chiou, J. C. and Farhat, C. (1991). A modular multibody analysis capability for high-precision, active control and real-time applications. *Int. J. Numer. Meth. Engng* **32,** 1767–1798.

Reissner, E. (1973). On one-dimensional large displacement finite-strain beam theory. *Stud. Appl. Math.* **52**(2), 87–95.

Simo, J. C. (1985). A finite strain beam formulation. The three-dimensional dynamic problem. Part I. *Comput. Meth. Appl. Mech. Engng* **49,** 55–70.

Simo, J. C. and Vu-Quoc, L. (1986a). On the dynamics of flexible beams under large overall motions—the plane case: Part I. *ASME J. Appl. Mech.* **53**(4), 849–854.

Simo, J. C. and Vu-Quoc, L. (1986b). On the dynamics of flexible beams under large overall motions—the plane case: Part II. *ASME J. Appl. Mech.* **53**(4), 855–863.

Simo, J. C. and Vu-Quoc, L. (1986c). Three-dimensional finite-strain rod model. Part II: Computational aspects. *Comput. Meth. Appl. Mech. Engng* **58**(1), 79–116.

Simo, J. C. and Vu-Quoc, L. (1988). On the dynamics in space of rods undergoing large motions—a geometrically-exact approach. *Comput. Meth. Appl. Mech. Engng* **66,** 125–161.

Simo, J. C. and Vu-Quoc, L. (1991). A geometrically-exact beam model incorporating shear and torsion warping deformation. *Int. J. Solids Struct.* **27**(3), 371–393.

Surana, K. S. and Sorem, R. M. (1989). Geometrically nonlinear formulation for three-dimensional curved beam elements with large rotations. *Int. J. Numer. Meth. Engng* **28,** 43–73.

Vu-Quoc, L. (1986). Dynamics of flexible structures performing large overall motions: a geometrically-nonlinear approach. Doctoral Dissertation, University of California, Berkeley.

APPENDIX A

To relate the different coordinate systems used in this formulation, several orthogonal transformations must be introduced. Considering one structural component they are as follows:

$$\mathbf{b}_{\alpha i} = \mathbf{C}_i \mathbf{a}_{\alpha i} \quad \mathbf{b}_{\alpha j} = \mathbf{C}_j \mathbf{a}_{\alpha j} \tag{A1}$$

$$\mathbf{a}_{\alpha i} = \mathbf{C}_i^0 \mathbf{g}_\alpha \quad \mathbf{a}_{\alpha j} = \mathbf{C}_j^0 \mathbf{g}_\alpha \tag{A2}$$

$$\mathbf{E}_{\alpha 0} = \mathbf{T}_i \mathbf{a}_{\alpha i} \quad \mathbf{E}_{\alpha 0} = \mathbf{T}_j \mathbf{a}_{\alpha j} \tag{A3}$$

$$\mathbf{E}_\alpha = \mathbf{H}\mathbf{E}_{\alpha 0} \quad \mathbf{e}_\alpha = \mathbf{C}_c \mathbf{E}_{\alpha 0} \tag{A4}$$

$$\mathbf{b}_{\alpha i} = \mathbf{C}_i \mathbf{T}_i^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{E}_\alpha = \mathbf{D}_i \mathbf{E}_\alpha \tag{A5}$$

$$\mathbf{b}_{\alpha j} = \mathbf{C}_j \mathbf{T}_j^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{E}_\alpha = \mathbf{D}_j \mathbf{E}_\alpha. \tag{A6}$$

Note that $\mathbf{C}_n^0$ and $\mathbf{T}_n$ are matrices that describe the initial orientation of the nodes and beam element cross-sections, respectively. These matrices are determined by using coordinate transformations on the initial coordinates. Also note that the rotation matrix $\mathbf{H}$, relating the initial and current orientations of the beam axis, is determined by using coordinate transformations on the current coordinates. $\mathbf{C}_e$, $\mathbf{C}_i$ and $\mathbf{C}_j$ are all functions of their respective rotation parameters.

Rotation matrix in terms of rotation parameters

Let us consider a problem in which finite rotations are described by the three modified Rodrigues parameters ϕ_1, ϕ_2 and ϕ_3. These parameters define a simple rotation relating two sets of orthogonal triads. For example, consider the rotation matrix $\mathbf{C}_e$ relating the triads $\mathbf{e}_\alpha$ and $\mathbf{E}_{\alpha 0}$. This rotation matrix can be written in terms of these parameters as follows:

$$\mathrm{C}_e = \frac{\begin{bmatrix} 1+\frac{1}{4}(\phi_1^2-\phi_2^2-\phi_3^2) & \frac{1}{2}\phi_1\phi_2-\phi_3 & \frac{1}{2}\phi_3\phi_1+\phi_2 \\ \frac{1}{2}\phi_1\phi_2+\phi_3 & 1+\frac{1}{4}(\phi_2^2-\phi_3^2-\phi_1^2 & \frac{1}{2}\phi_2\phi_3-\phi_1 \\ \frac{1}{2}\phi_3\phi_1-\phi_2 & \frac{1}{2}\phi_2\phi_3+\phi_1 & 1+\frac{1}{4}(\phi_3^2-\phi_1^2-\phi_2^2) \end{bmatrix}}{1+\frac{1}{4}(\phi_1^2+\phi_2^2+\phi_3^2)}. \tag{A7}$$